An Effective Dual Abelian-Higgs Model from SU(2) Yang-Mills theory via Connection Decomposition

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Abstract

It has long argued that confinement in non-Abelian gauge theories, such as QCD, can be account for by analogy with typed II superconductivity. In this paper, we show that it is possible to arrive at an effective dual Abelian-Higgs model, the dual and relativistic version of Ginzburg-Landau model for superconductor, from $SU(2)$ Yang-Mills theory based on the Faddeev-Niemi connection decomposition and the order-disorder assumptions for the gauge field. The implication of these assumptions is discussed and role of the resulted scalar field is analyzed associated with the "electric-magnetic" duality and theory vacuum. It is shown that the mass generation of the gauge vector field can arise from quantum fluctuation of the coset basis variable $\partial\mathbf{n}$, and the mass of the "electric" field is approximately equal to that of the scalar particle. A generalized dual London equation with topologically quantized singular vortices is derived for the static "electric" field from the our dual model.

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Key Words Dual Abelian-Higgs model, Yang-Mills theory, Connection decomposition, Dual London equation

1 Introduction

Three decades ago, the dual-superconductor(DS) picture of confinement in quantum chromodynamics (QCD) was proposed by Nambu, Mandelstam and 't Hooft [1, 2], in an analogy to type II superconductivity. In DS picture, color and quarks are confined within hadrons due to the dual Meissner effect. This picture was further elaborated by idea of Abelian projection [3] that QCD can be reduced, by partially-fixing gauge to Maximal Abelian(MA) gauge, to an Abelian

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gauge theory with magnetic charges forming condensate. This idea was confirmed by lattice calculations (see [4, 5, 6] for a review) for long-distance gluondynamics.

On the other hands, an proposal was put forward by 't Hooft[7] that an gauge-invariants scalar kernel $Z(\phi)$ should exist in the effective model for confinement as relicts of the infrared counter term. However, the its implication in framework of Yang-Mills(YM) theory remains unknown, even if the color confinement is believed to take place in the pure gluondynamics[8]. By deriving an effective Abelian-Higgs(AH) model from SU(2) YM theory based on a series of assumptions, one argument [9] suggests that ϕ in AH model can arise from off-diagonal gauge bosons. The assumptions include the order-stochastic assumptions for the diagonal and the off-diagonal parts of gauge vector field, the de-correlation assumption between the two parts and simple-mode approximation for the correlation function of off-diagonal field. However, in such an approach, the connections of the scalar to the magnetic charges and "electronic-magnetic" duality was not seen clearly.

The purpose of this paper is to point out that it is possible to derive an effective dual AH model starting from $SU(2)$ YM theory, based on the three assumptions that is similar to the ideas in Ref. [9]. In this model, the scalar ϕ comes from off-diagonal gauge degrees of freedom and the gauge-invariants scalar kernel $Z(\phi)$ appears as an effective magnetic-media factor for theory vacuum, whose the vacuum expectation value(vev.) can provide a scalar potential of Mexico-hat form. This is done by reformulating YM theory via the connection decomposition (CD)[10, 11, 12], and then use the order-disorder assumptions for the new variables in CD. We also use the de-correlation assumptions between the dual variables and three local basis in iso-space. The mass generation of the Abelian field can arise from quantum fluctuation of the off-diagonal basis $\partial\mathbf{n}$, and is approximately equal to that of the scalar particle. A generalized dual London equation with topologically quantized singular vortices was derived for the static "electric" field from the dual AH model.

2   Abelian projection in terms of connection decomposition

In the approach of CD[10, 11, 12], the separating the infrared variables from YM connection can be done by introducing infrared unit order-parameter field $\mathbf{n}(x)$. Based on CD, the vacuum structure of the YM theory and gluonball spectrum are studied, associated with knotted-vortex excitations[13, 14]. Here, we study DS picture in YM theory from the viewpoint of CD approach.

We consider $SU(2)$ YM theory where connection("gluon" field) $\mathbf{A}_\mu = A_\mu^a \tau^a$ ($\tau^a = \sigma^a/2, a = 1, 2, 3$) describes 6 transverse ultraviolet degrees of freedom. To parameterize $\mathbf{A}_\mu$ in terms of new variables, we invoke an unit iso-vector $\mathbf{n}$. Solving $\mathbf{A}_\mu$ from $D_\mu \mathbf{n} - \partial_\mu \mathbf{n} = g\mathbf{A}_\mu \times \mathbf{n}$, where g is coupling constant, one gets[10]

$$\mathbf{A}_\mu = A_\mu \mathbf{n} + g^{-1}\partial_\mu \mathbf{n} \times \mathbf{n} + \mathbf{b}_\mu \tag{1}$$

where $A_\mu \equiv \mathbf{A}_\mu \cdot \mathbf{n}$ transforms as an Abelian connection for $U(1)$ rotation $U(\alpha) = e^{i\alpha n^a \tau^a}$ round the iso-direction $\mathbf{n}$ ($A_\mu \to A_\mu + \partial_\mu \alpha/g$) and $\mathbf{b}_\mu = g^{-1}\mathbf{n} \times D_\mu(\mathbf{A}_\mu)\mathbf{n}$ is $SU(2)$ covariant. Here,

the Abelian part $A_\mu\mathbf{n}$ in (1) corresponds to Abelian subgroup $H = U(1)$, while the non-Abelian gluon parts $\mathbf{C}_\mu = g^{-1}\partial_\mu\mathbf{n} \times \mathbf{n}$ and $\mathbf{b}_\mu = b_\mu^a\tau^a$, both of which are orthogonal to $\mathbf{n}$, correspond to coset group $SU(2)/H$. We note that (1) can be true variable change [15] if one takes $\mathbf{b}_\mu$ itself as a gauge vector field and further imposes two constraints on $\mathbf{b}_\mu$. This is necessary for getting marginal contribution to the final effective action but we do not consider such a contribution in this paper.

The fact that $\mathbf{C}_\mu$ does not depend upon the original degrees of freedom $\mathbf{A}_\mu$ implies $\mathbf{A}_\mu$ has intrinsic structure. This idea is firstly due to the work on multi-monopoles[10] and has been generalized to the $SO(N)$-connection case[16, 17] as well as the spinorial-decomposition case[18, 19]. Further decomposing $\mathbf{b}_\mu$ in terms of the local basis $\{\partial_\mu\mathbf{n}, \partial_\mu\mathbf{n} \times \mathbf{n}\}$ of the internal coset space $SU(2)/H$, one find CD [12] for $SU(2)$ connection

$$\mathbf{A}_\mu = A_\mu\mathbf{n} + \mathbf{C}_\mu + g^{-1}\phi_1\partial_\mu\mathbf{n} + g^{-1}\phi_2\partial_\mu\mathbf{n} \times \mathbf{n}, \tag{2}$$

in which A_μ (and $\mathbf{A}_\mu$) has dimension of mass, $\mathbf{n}$, the scalars ϕ_1 and ϕ_2 of unit.

The transformation role of the new variables in (2) under the gauge rotation $U(\alpha)$ can be found by requiring the CD (2) to be covariant under $U(\alpha)$. Clearly, $\mathbf{C}_\mu$ is intact under $U(\alpha)$. The transformation role of ϕ_1 and ϕ_2 can be given by the $U(\alpha)$-covariance of $\mathbf{b}_\mu$. In fact, one finds

$$\begin{aligned}(b_\mu^a\tau^a)^U &= g^{-1}e^{i\alpha\mathbf{n}\cdot\boldsymbol{\tau}}(\phi_1\partial_\mu n^a\tau^a + \phi_2\epsilon^{abc}\partial_\mu n^b n^c\tau^a)e^{-i\alpha\mathbf{n}\cdot\boldsymbol{\tau}},\\ &= g^{-1}(\phi_1 - \alpha\phi_2)\partial_\mu n^a\tau^a + g^{-1}(\phi_2 + \alpha\phi_1)(\partial_\mu\mathbf{n} \times \mathbf{n})^a\tau^a,\end{aligned}$$

which implies $\delta\phi_1 = -\alpha\phi_2$ and $\delta\phi_2 = \alpha\phi_1$, or $\delta(\phi_1 + i\phi_2) = i\alpha(\phi_1 + i\phi_2)$. Thus, the complex variables $\phi = \phi_1 + i\phi_2$ transforms as a charged complex scalar:

$$\phi \rightarrow \phi e^{i\alpha}.$$

In comparison with 12 field components of $\mathbf{A}_\mu$, the new variables (A_μ, n^a, ϕ) in (2) has $8(= 4 + 2 + 2)$ degrees of freedom, making CD (2) a singular variable change. The CD then defines a singular gauge-fixing in which $A_\mu\mathbf{n}$ is the unfixed diagonal part of the vector field $\mathbf{A}_\mu$ and the other terms in RHS of (2) are non-Abelian(off-diagonal) components. The singularities in CD (2) are caused by the difference between two group manifolds $SU(2)$ and H, and is determined by the singularities in $\mathbf{n}(x)$ [15]. The singularities in $\mathbf{n}(x)$ is due to the global parameterization of a gauge group with nontrivial topology in terms of local basis, and they responds to magnetic monopoles configuration, as shown below.

In fact, if one chooses $\mathbf{n}$ as hedgehog configuration $\mathbf{n} = \mathbf{x}/r$ then $\mathbf{C}_\mu$ becomes the Wu-Yang potential [20] for non-Abelian magnetic monopole:$A_i^a = \epsilon_{aic}x^c/r^2$. The field strength $\mathbf{C}_{\mu\nu} = \partial_\mu\mathbf{C}_\nu - \partial_\nu\mathbf{C}_\mu$ for $\mathbf{C}_\mu$ can be written as $\mathbf{C}_{\mu\nu} \equiv B_{\mu\nu}\mathbf{n}$ where

$$B_{\mu\nu} \equiv -g^{-1}\mathbf{n} \cdot (\partial_\mu\mathbf{n} \times \partial_\nu\mathbf{n}) \tag{3}$$

stands for the magnetic field strength. One can identify the (projected) magnetic potential C_μ by definition $B_{\mu\nu} \equiv \partial_\mu C_\nu - \partial_\nu C_\mu$, where the parametrization of C_μ can not be given

3

globally. This can be seen by parameterizing $\mathbf{n}$ in terms of spherical coordinates as $\mathbf{n} = (\sin\gamma\cos\beta, \sin\gamma\sin\beta, \cos\gamma)$. One get from (3)

$$
\begin{aligned}
B_{\mu\nu} &= -g^{-1}\sin\gamma(\partial_\mu\gamma\partial_\nu\beta - \partial_\nu\gamma\partial_\mu\beta) \\
C_\mu &= g^{-1}(\cos\gamma\partial_\mu\beta \pm \partial_\mu\alpha),
\end{aligned}
$$

which takes the form of the $SU(2)$ magnetic configuration. C_μ is not uniquely defined and has a gauge freedom of $U(1)$ transformation $(C_\mu \to C_\mu + g^{-1}\partial_\mu\alpha')$, corresponding to the rotation $U(\alpha')$ round $\mathbf{n}$. This $U(1)$ covariance is happened to hold simultaneously for Abelian part A_μ. By setting $\mathbf{n}$ along σ^3, one sees that choosing a local direction $\mathbf{n}(x)$ at each point x ensures the covariance of CD for $U(\alpha')$-rotation, but not for the rotation round σ^1 or σ^2. The singularities of $\mathbf{n}(x)$ occurs at points where the orientation of $\mathbf{n}(x)$ is not defined due to the global reparametrization of the field variables in terms of the local basis $(\mathbf{n}, d\mathbf{n}, d\mathbf{n}\times d\mathbf{n})$.

By comparison CD (2) with the global decomposition

$$
\mathbf{A}_\mu = A_\mu^3\tau^3 + A_\mu^1\tau^1 + A_\mu^2\tau^2, \tag{4}
$$

one finds that there are the local correspondences between $A_\mu\mathbf{n}$ and diagonal part $A_\mu^3\tau^3$, and (ϕ_1, ϕ_2)-terms and $(A_\mu^1\tau^1, A_\mu^2\tau^2)$. However, no correspondence exists in (4) for the potential $\mathbf{C}_\mu$. If we parameterize the SU(2) matrix in terms of Euler angles $(\alpha(x), \beta(x), \gamma(x))$:

$$
U(x) = e^{i\beta\sigma^3/2}e^{i\gamma\sigma^2/2}e^{i\alpha\sigma^3/2} = \begin{pmatrix} e^{i(\beta+\alpha)/2}\cos(\gamma/2) & -e^{i(\beta-\alpha)/2}\sin(\gamma/2) \\ e^{-i(\beta-\alpha)/2}\sin(\gamma/2) & e^{-i(\beta+\alpha)/2}\cos(\gamma/2) \end{pmatrix}
$$

one finds

$$
\begin{aligned}
C_\mu(x) &= tr\left(\sigma^3\frac{i}{g}U(x)\partial_\mu U^\dagger(x)\right) \tag{5} \\
n^a(x) &= 2Tr\left(\sigma^a U^\dagger(x)\tau^3 U(x)\right) \tag{6}
\end{aligned}
$$

i.e., the magnetic potential C_μ originates from the pure gauge $iU\partial_\mu U^\dagger/g$. Therefore, when we choose $U(x)$ to be the singular transformation mapping the global basis $\{\tau^{1\sim3}\}$ to the local basis $\{\mathbf{n}, \partial_\mu\mathbf{n}, \partial_\mu\mathbf{n}\times\mathbf{n}\}$ and set $\mathbf{n}(\infty) = \mathbf{n}_0$(constant unit vector) we get one example of Abelian projection, where the maximal Abelian subgroup H responds to the gauge rotation round $\mathbf{n}$. This implies that CD makes the basis dynamic and contains the magnetic variable as a topological degree of freedom. The monopole occurs at the singularities of the transformation $U(x)$ or iso-vector $\mathbf{n}(x)$, as can be seen from (5) and (6).

With (2), one finds the gauge field strength:

$$
\begin{aligned}
\mathbf{G}_{\mu\nu} &= \mathbf{n}[F_{\mu\nu} + Z(\phi)B_{\mu\nu}] + g^{-1}\left[\nabla^\mu\phi\mathbf{n}_\nu - \nabla_\nu\phi\mathbf{n}_\mu\right] \\
&\quad + \frac{1}{2g}\left[\nabla_\mu\phi\mathbf{n}_\nu - \nabla_\nu\phi\mathbf{n}_\mu\right] \tag{7}
\end{aligned}
$$

where $F_{\mu\nu} \equiv \partial_\mu A_v - \partial_v A_\mu$, $Z(\phi) = 1 - |\phi|^2$ and $\nabla_\mu\phi \equiv \nabla_\mu\phi_1 + i\nabla_\mu\phi_2 = (\partial_\mu - igA_\mu)\phi$ is $U(1)$ covariant derivative where $\nabla_\mu\phi_1 \equiv \partial_\mu\phi_1 + gA_\mu\phi_2$ and $\nabla_\mu\phi_2 \equiv \partial_\mu\phi_2 - gA_\mu\phi_1$. Here, $\mathbf{n}_\mu = \partial_\mu\mathbf{n} - i\partial_\mu\mathbf{n}\times\mathbf{n}$ and $h.c.$ stands for Hermite conjugation.

4

It is suggestive to consider the Abelian components of the full gauge field $\mathbf{G}_{\mu\nu}$

$$\mathbf{G}_{\mu\nu} \quad \rightarrow \quad \mathbf{n}[F_{\mu\nu} + H_{\mu\nu}], \tag{8}$$

$$H_{\mu\nu} \quad : \quad = Z(\phi)B_{\mu\nu}.$$

This gives a hint for "electric-magnetic" duality $(A_\mu \leftrightarrow C_\mu,\ F_{\mu\nu} \leftrightarrow H_{\mu\nu}$) in an effective "magnetic media" described by the off-diagonal variable ϕ. Similar media-like factor $Z(\phi)$ is introduced phenomenologically by 't Hooft[7] to account for the QCD vacuum. The Eq.(8) can also be reached by taking large-g limit, which transforms the standard YM theory into an pure Abelian gauge theory with "electric" charge as well as "magnetic" charge. One can see from (8) that owing to the off-diagonal field the original "electric-magnetic" duality given by 't Hooft tensor $f_{\mu\nu} = F_{\mu\nu} + B_{\mu\nu}$ becomes that in effective "magnetic media".

3   Order-disordered assumption for gauge connection

To approach the infrared YM theory non-perturbatively, we will use three assumptions about the gauge field variables, which is similar to, but not same with that presented in Ref. [9]. We argued that these assumptions is valid approximately in far-infrared regime. The three main assumptions are listed as below:

(1). The off-diagonal basis $\partial_\mu \mathbf{n}$ in CD becomes stochastic while $\mathbf{n}$ is ordered in quantum YM theory. That is, one has following assumption on the vev:

$$\langle \partial_\mu n^a(x) \rangle \quad = \quad 0,\ \text{and}\ \langle \partial_\mu n^a(x) \partial^\nu n^b(x) \rangle \neq 0, \tag{9}$$

$$\text{only if}\ \mu \quad = \quad \nu, a = b$$

and $\langle \partial_\mu n^a n^b \rangle = 0$ even if $a = b$. This is consistent with the gauge, Lorentz and parity invariance of the ground state. The nonvanishing vev. is looked as a consequence of the quantum fluctuation of $\partial_\mu \mathbf{n}$. Eq.(9) means $\langle \partial^\mu n^a \partial_\nu n^a \rangle \propto \delta^\mu_\nu$.

(2). The Abelian field A_μ behaves as classical variable, and the dual variables (A_μ, ϕ) are de-correlated with basis $\mathbf{n}$ and $\partial_\mu \mathbf{n}$. That means

$$\langle F(A_\mu, \phi) P(\mathbf{n}, \partial \mathbf{n}) \rangle = \langle F(A_\mu, \phi) \rangle \langle P(\mathbf{n}, \partial \mathbf{n}) \rangle, \tag{10}$$

where F and P are any functions of the involved variables. The classical behavior of A_μ means that $\langle f(A_\mu) \rangle \approx f(A_\mu)$ being a good approximation for any function f of A_μ. This has close analogy to the semi-classical treatment of the radiation field, where the radiation field behaves as classical variable while the matter degrees of freedom are taken to be the quantum operators.

(3). After quantization of theory, the complex variable $\phi^*(x)$ is taken to be a field operator $\phi^\dagger$ of creating a charged scalar particle at x, and $\phi(x)$ to be the the corresponding operator annihilating that the particle. In the vacuum state, ϕ shows off-diagonal long range order (ODLRO), which can be explicitly written as

$$\langle \phi(x) \phi^\dagger(y) \rangle = \Phi(x) \Phi^*(y), \text{for}\ x_0 > y_0. \tag{11}$$

One may doubt the justification of above three assumptions. It is true, to my knowledge, that no direct evidence for these assumptions exists. However, the indication that the off-diagonal gluon amplitude is strongly suppressed and off-diagonal gluon phase shows strong randomness is indeed observed in lattice simulation [21, 22]. Furthermore, the order-disorder transition[23, 24] is also predicted numerically in which an operator of creating magnetically-charged particle similar to $\phi^\dagger$ is shown to be the order parameter for this transition. The same results was also seen in the numerical analysis of the specific heat and of the chiral order parameter at the chiral transition[25]. We note that the ODLRO assumption for ϕ in (3) does not contradict with the randomness of the off-diagonal gauge potential in the sense that the later is caused by the randomness of the local basis field $\{\partial_\mu \mathbf{n}, \partial_\mu \mathbf{n} \times \mathbf{n}\}$ of the coset group $SU(2)/H$. One can ready verify by Eqs. (9) and (10) that $\mathbf{b}_\mu$ is stochastic: $\langle \mathbf{b}_\mu \rangle = 0$ but $\langle (\mathbf{b}_\mu)^2 \rangle \neq 0$. In addition, the classical behavior of A_μ was always assumed in DS analysis (see [26] and references therein) of the infrared QCD and was predicted in the lattice simulations[23, 21, 22].

The most subtle ansatz may be the assumption (10). It implies that the "magnetic" field does not couple with AH variables (A_μ, ϕ) at the classical level. When combining with the ansatz (9), this implies the variables $\mathbf{n}$ in (2) forms the background field in with contrast (A_μ, ϕ), as we will shown in the next section. We will see there that the ansatz (10) is necessary for us to derive the dual AH model and the combining of three assumptions will lead to the Bogomolnyi limit[27] $m_\Phi \approx m_A$, being consistent with numerical result [28].

3.1   Effective dual Abelian-Higgs action

Putting (7) into the YM Lagrangian $\mathfrak{L} = -\mathbf{G}_{\mu\nu}^2/4$, one gets

$$\mathfrak{L}_{dual} = -\frac{1}{4}F_{\mu\nu}^2 + \mathfrak{L}_V + \mathfrak{L}_{FB} + \mathfrak{L}_D, \tag{12}$$

where

$$\mathfrak{L}_V = -\frac{1}{4}Z(\phi)^2 B_{\mu\nu}^2$$

$$\mathfrak{L}_{FB} = -\frac{1}{2}Z(\phi)F_{\mu\nu}B^{\mu\nu}$$

$$\mathfrak{L}_D = -\frac{1}{2g^2}(n_{\mu\nu} - igB_{\mu\nu})(\nabla^\mu\phi)^\dagger \nabla^\nu\phi$$

Here, $n_{\mu\nu} \equiv \eta_{\mu\nu}(\partial_\rho \mathbf{n})^2 - \partial_\mu \mathbf{n} \cdot \partial_\nu \mathbf{n}$ and ϕ appears as a charged field minimally coupled with 'electric' field A_μ by strength g.

We look all variables in (2) as quantum operators. Since the ground state is both the (relativistic and gauge) rotation and translation invariant, the stochastic assumption (1) for $\partial \mathbf{n}$ implies

$$\langle (\partial_x \mathbf{n})^2 \rangle = \langle (\partial_y \mathbf{n})^2 \rangle = \langle (\partial_z \mathbf{n})^2 \rangle = (\partial_0 \mathbf{n})^2 = m^2 \tag{13}$$

$$\langle (\partial_\mu n^a)^2 \rangle = \frac{1}{3}\langle (\partial_\mu \mathbf{n})^2 \rangle = -\frac{2}{3}m^2, \text{for fixed } a = 1, 2, 3. \tag{14}$$

in which m is a mass scale of the $\partial\mathbf{n}$-fluctuation. This gives

$$\langle(\partial\mathbf{n})^2\rangle := \langle\partial^\mu n^a(x)\partial_\mu n^a(x)\rangle = -2m^2. \tag{15}$$

From (9), (13) and (15), one has

$$\langle n^\mu_\nu\rangle = \delta^\mu_\nu\langle(\partial\mathbf{n})^2\rangle - \delta^\mu_\nu\langle(\partial_0\mathbf{n})^2\rangle = -3\delta^\mu_\nu m^2 \tag{16}$$

Using Wick theorem and the assumption (1), one can show

$$\begin{aligned}
\langle B_{\mu\nu}\rangle &= g^{-1}\epsilon^{abc}\lim_{\varepsilon^\mu\to 0}\partial^{x_2}_\mu\partial^{x_3}_\nu\langle n^a(x_3)n^b(x_2)n^c(x_1)\rangle \\
&= g^{-1}\epsilon^{abc}\lim_{x_3\to x}\langle n^a(x_3)\rangle\langle\partial_\mu n^b(x_2)\partial_\nu n^c(x_1)\rangle|_{x_2=x_1=x} \\
&\propto \epsilon^{abc}\langle\partial_\mu n^b(x)\partial_\nu n^c(x)\rangle = 0,
\end{aligned}$$

where $\varepsilon^\mu \equiv max_{i,j}\|(x_i)^\mu - (x_j)^\mu\|$ is four small positive parameters $(i,j = 1,2,3)$ and the time-order was assumed so that $(x)^0 < (x_1)^0 < (x_2)^0 < (x_3)^0 < (x)_0 + \varepsilon^0$ before taking limit.

Including the full off-diagonal variables and applying the assumptions (1)$\sim$(3) to the reformulated YM Lagrangian (12), one can calculate the vacuum average of the Lagrangians as below. First, one has

$$\begin{aligned}
\langle\mathfrak{L}_V\rangle &= -(\lambda/4)\left\langle 1 + (\phi^\dagger\phi)^2 - 2\phi^\dagger\phi\right\rangle \\
&= -(\lambda/4)\left(1 + 2|\Phi^*\Phi|^2 - 2\Phi^*\Phi\right) \\
&= -(\lambda/2)\left[(|\Phi|^2 - 1/2)^2 + 1/4\right]
\end{aligned} \tag{17}$$

where $\lambda \equiv \langle B^2_{\mu\nu}\rangle$ is positive scale and with dimension of 4. We have used the assumption (2) and (3), the Bose symmetry of the scalar field and the Wick theorem:

$$\begin{aligned}
\langle\phi^\dagger\phi\phi^\dagger\phi\rangle|_x &= \lim_{\varepsilon^\mu\to 0}\langle\phi^\dagger_1\phi_2\phi^\dagger_3\phi_4\rangle \\
&= \lim_{\varepsilon^\mu\to 0}\{\langle\phi_4\phi^\dagger_3\rangle\langle\phi_2\phi^\dagger_1\rangle + \langle\phi_4\phi^\dagger_1\rangle\langle\phi_3\phi^\dagger_2\rangle + \langle\phi^\dagger_1\phi^\dagger_3\rangle\langle\phi_2\phi_4\rangle\} \\
&= 2\Phi(x)\Phi^*(x)\Phi(x)\Phi^*(x).
\end{aligned}$$

Here, $\phi_i \equiv \phi(x_i)$ and ε^μ are the maximal norm of $(x_i)^\mu - (x_j)^\mu$, where $i,j = 1\sim 4$. The time-order is assumed so that $(x)^0 < (x_1)^0 < \cdots < (x_4)^0 < (x)_0 + \varepsilon^0$. Moreover, $\langle\phi^\dagger_1\phi^\dagger_3\rangle = \langle\phi_2\phi_4\rangle = 0$ for $x_1 = x_3$, $x_2 = x_4$ since in the Fock state of vacuum only paired products of ϕ and $\phi^\dagger$ has nonvanishing vev.. Furthermore,

$$\begin{aligned}
\langle[\nabla^\mu\phi(x)]^\dagger\nabla_\mu\phi(x)\rangle &= \langle(\partial_\mu\phi^\dagger(x)\partial^\mu\phi(x) + igA^\mu(x)\phi^\dagger(x)\partial_\mu\phi(x) \\
&\quad -igA_\mu(x)\partial^\mu\phi^\dagger(x)\phi(x) + g^2A^\mu(x)A_\mu(x)\phi^\dagger(x)\phi(x)\rangle \\
&= \lim_{y\to x^-}[\partial^\mu_x\partial_{\mu y} + igA^\mu\partial_{\mu x} - igA_\mu\partial^{\mu y} + g^2A^\mu A_\mu]\left\langle\phi(x)\phi^\dagger(y)\right\rangle,
\end{aligned}$$

which leads to

$$\begin{aligned}
\langle[\nabla^\mu\phi]^\dagger\nabla_\mu\phi\rangle &= \partial^\mu\Phi^*(x)\partial_\mu\Phi(x) + igA^\mu(x)\partial_\mu\Phi(x)\Phi^*(x) \\
&\quad -igA_\mu(x)\Phi(x)\partial^\mu\Phi^*(x) + g^2A^\mu(x)A_\mu(x)\Phi(x)\Phi^*(x) \\
&= [\nabla_\mu\Phi(x)]^*\nabla^\mu\Phi(x)
\end{aligned} \tag{18}$$

7

Combining with (16) and the relation $\langle B_{\mu\nu}(\nabla^\mu\phi)^\dagger\nabla^\nu\phi\rangle = 0$, this gives

$$\langle \mathcal{L}_D\rangle = \frac{3m^2}{2g^2}|(\partial_\mu - igA_\mu)\Phi|^2 \tag{19}$$

Noticing that $\langle \mathcal{L}_{FB}\rangle \propto \langle B^{\mu\nu}\rangle = 0$ due to the assumptions (2), and putting (17), (19) into (12), one gets

$$\begin{aligned}
\mathcal{L}^{eff} &= -\frac{1}{4}F_{\mu\nu}^2 + \frac{3m^2}{g^2}|\nabla^\mu\Phi|^2 \\
&\quad -\frac{\lambda}{2}(|\Phi|^2 - \frac{1}{2})^2 - \frac{\lambda}{8}
\end{aligned} \tag{20}$$

Since $\lambda > 0$ has dimension of 4 and is proportional to g^{-2} one can rewrite it as $\lambda = m^{*4}/g^2$, in which m^{*4} is a positive mass scale determined by:

$$m^{*4} = \left\langle (\mathbf{n}, d\mathbf{n}\times d\mathbf{n})^2\right\rangle. \tag{21}$$

By scaling Φ into that with dimension of mass

$$\sqrt{\frac{3}{2}}\frac{m}{g}\Phi(x)\to\Phi(x), \tag{22}$$

and ignoring an additive constant in (20), we arrive at the effective dual AH Lagrangian

$$\mathcal{L}^{eff} = -\frac{1}{4}F_{\mu\nu}^2 + |(\partial_\mu - igA_\mu)\Phi|^2 - V(\Phi) \tag{23}$$

where the potential $V(\Phi)$ of the scalar Φ has the Mexico-hat form

$$\begin{aligned}
V(\Phi) &= \frac{\widetilde{\lambda}}{4}(|\Phi|^2 - \mu^2)^2. \\
\widetilde{\lambda} &= \frac{8g^2}{9}(\frac{m^*}{m})^4, \mu = \frac{\sqrt{3}m}{2g} > 0.
\end{aligned} \tag{24}$$

We can determine m^* in (21) by using the assumption (1). Noticing the relation (9) and using Wick theorem, we find

$$\begin{aligned}
m^{*4} &= \epsilon_{abc}\epsilon_{mkl}[\langle\partial_\mu n^b\partial^\mu n^k\partial_\nu n^c\partial^\nu n^l n^a n^m\rangle] \\
&= \epsilon_{abc}\epsilon_{mkl}[\langle\partial_\mu n^b\partial^\mu n^k\rangle\langle\partial_\nu n^c\partial^\nu n^l\rangle\langle n^a n^m\rangle + \langle\partial_\mu n^b\partial^\nu n^l\rangle\langle\partial_\nu n^c\partial^\mu n^k\rangle\langle n^a n^m\rangle \\
&\quad + \langle\partial_\mu n^b n^m\rangle\langle\partial_\nu n^c\partial^\nu n^l\rangle\langle\partial^\mu n^k n^a\rangle + \cdots] \\
&= \epsilon_{abc}\epsilon_{mkl}\delta^{bk}\delta^{cl}\langle(\partial_\mu n^1)^2\rangle^2\langle n^a n^m\rangle \\
&= 2!(\frac{-2m^2}{3})^2\delta^{am}\langle n^a n^m\rangle \\
&= \frac{8}{9}m^4.
\end{aligned}$$

where we have used the relation (14). With (13), one has $\langle(\nabla\mathbf{n})^2\rangle = 3m^2$. Thus

$$\begin{aligned}
\widetilde{\lambda} &= (\frac{8g}{9})^2 \\
m &= \langle(\nabla\mathbf{n})^2\rangle^{1/2}/\sqrt{3}
\end{aligned} \tag{25}$$

It follows that the mass of the complex scalar Φ is

$$m_\Phi = \sqrt{\tilde{\lambda}}\mu = \frac{4\sqrt{3}}{9}m,$$

The model (23) is well known as the dual AH model in the original dual-superconductor mechanism[1, 29] for the confining phase of QCD, as an analogy to the Ginzburg-Landou model for superconductor. The dual Higgs mechanism for the model (23) will enable A_μ to acquires a mass.

The Hamiltonian with respect to the Lagrangian (23) is

$$
\begin{aligned}
\mathcal{H} &= \int d^3x \left[|D^0\Phi|^2 + |\vec{D}\Phi|^2 + \frac{1}{2}(\partial_t\vec{A} + \nabla A^0)^2 \right. \\
&\quad \left. + \frac{1}{2}\vec{B}^2 + V(\Phi) \right],
\end{aligned}
$$

which has a lowest energy solution $A_\mu(x) = 0, \Phi(x) = \mu, (\mu > 0)$. Here, $\vec{B} := \nabla \times \vec{A}$ and the global phase factor has canceled for simplicity. Making the shift $\Phi(x) = \mu + \eta(x)$ enables us to rewrite Lagrangian as

$$\mathfrak{L}^{eff} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{3m^2}{4}A_\mu A^\mu + (\partial_\mu\eta)^2 - V(\eta), \tag{26}$$

where η can be taken to be a real scalar by choosing unitary gauge, and it has potential

$$V(\eta) = \frac{16g^2}{81}\eta^2[\eta + 2\mu]^2. \tag{27}$$

Clearly, $V(\eta)$ favors the zero vev. of η. Thus, the Higgs mechanism leads A_μ to acquires a mass

$$m_A = \sqrt{\frac{3}{2}}m \tag{28}$$

and the complex scalar Φ becomes into a real scalar particle η with mass

$$m_\eta = m_\Phi = \frac{4\sqrt{3}}{9}m, \tag{29}$$

as can be seen in (26) and (27). Then, one has

$$\frac{m_A}{m_\Phi} \approx \frac{3}{2}$$

We see here that, as a consequences of order-disorder and de-correlation transition of the variables $(A_\mu, \mathbf{n}, \phi)$ in infrared limit, the effective theory of YM theory becomes dual AH model, in which both the Abelian "electric" field A_μ and the scalar Φ(and η) acquire the masses $\sim m$. The mass scale m arises from the the quantum fluctuation $\langle(\nabla\mathbf{n})^2\rangle^{1/2}$ of the stochastic field $\nabla\mathbf{n}$ and has the same order with m_η and m_A, though the determination of this mass scale quite nontrivial. This is in agreement with the the lattice simulation[28] where the Bogomolnyi limit[27] $m_\Phi \approx m_A$ for AH model is predicted by fitting the SU(2) Lattice gauge theory[32], as is observed in many Lattice QCD calculation[33].

One can also see that the effective potential $V(\Phi)$ which favors the SSB of the AH model (23) is developed from the averaged media-factor $\langle Z(\phi) \rangle$ and the resulted masses (m_η, m_A) for scalar and "electric" field do not depend explicitly on the coupling g, though the form of potential $V(\eta)$ does. This favors the idea of DS picture that the YM vacuum is of the "magnetic" type in far-infrared limit[1, 29]. A similar proof[30] is also proposed recently that the SU(2) YM theory can reduce to a two-band dual superconductor with an interband Josephson coupling.

3.2   Generalized Dual London equation

There are much evidences that vortices can be responsible for color confinement:Vortex configurations reproduce a great number of the asymptotic force between a static quark-antiquark pair (see [4] for a review). As a key ingredient in DS picture of confinement, such a topological configuration was recently shown [31] to be necessary in the light of Gribov-Zwanziger's confinement condition.

Here, we show that our effective AH model (23) can give rise to a dual generalized London's equation for Abelian "electric" field A_μ, which allows the Abrikosov-Nielsen-Olesen(ANO) vortex-string of the "electric" field at the zero of Φ. The equation of motion of the model (23) is

$$
\begin{aligned}
\partial_\nu F^{\mu\nu} &= -ig(\Phi^* \partial^\mu \Phi - \partial^\mu \Phi^* \Phi) - 2g^2 |\Phi|^2 A^\mu \\
\nabla_\mu \nabla^\mu \Phi &= -\frac{\widetilde{\lambda}}{2}[|\Phi|^2 - \mu^2]\Phi,
\end{aligned}
\tag{30}
$$

where the Lorenz gauge $(\partial_\mu A^\mu = 0)$ is used.

It is suggestive to first consider the uniform solution of Φ to Eq.(30). It is

$$
\begin{aligned}
\Phi &\approx \mu, \\
\partial_\nu F^{\mu\nu} &= j^\mu = -m_A^2 A^\mu.
\end{aligned}
\tag{31}
$$

We note that the second equation can also be obtained by taking the large-g limit in (30), where $\Phi(x) \sim g^{-1}$, as seen in (22). The second equation of Eq.(31) is known as London's relation, which is valid only in the uniform or slow-varying situation of $\Phi(x)$. The parameter m_A, given by (28), is the mass scale responsible for dual Meissner effect, and its inverse $\lambda_L = 1/m_A \sim m^{-1}$ determines the transverse dimensions of the "electric" field A_μ penetrating into the YM vacuum. This length scale is about 0.95fm for the data $m_A = 1.3$ Gev given in Ref [28]. For finite g, the numerical solutions for "electric" field and Φ are studied in Ref. [32, 28]. Below, we presents pure topological argument that the ANO vortex must exist at the classical level as "electric" vortex line by taking the zero of Φ into account.

We consider the static case of Eq.(30). Instead of writing $\Phi = |\Phi|e^{iS(x)}$ which enables $\vec{V} := -i\Phi^* \overleftrightarrow{\nabla} \Phi/(2|\Phi|^2)$ to take the form of velocity potential ($\vec{V} \propto \nabla S(x)$), we write Φ as $\Phi = \Phi_1 + i\Phi_2$, with real scalars Φ_1 and Φ_2 forming a vector field $\vec{\Phi} := (\Phi^1, \Phi^2)$. Denote its

direction field as $N^i = \Phi^i/|\Phi|, (i = 1.2)$ and rewrite $\vec{V}$ as $\vec{V} = \epsilon_{ij}N^i\nabla N^j$. Then, the vorticity of $\vec{V}$ becomes

$$
\begin{aligned}
(\nabla \times \vec{V})^a &= \epsilon^{abc}\epsilon_{ij}\partial_b[N^i\partial_c N^j] \\
&= \epsilon^{abc}\epsilon_{ij}\partial_b[\frac{\Phi^i}{|\Phi|^2}]\partial_c\Phi^j \\
&= \epsilon^{abc}\epsilon_{ij}[\frac{\partial^2}{\partial\Phi^i\partial\Phi^k}\ln|\Phi|]\partial_b\Phi^k\partial_c\Phi^j \\
&= \triangle_\Phi \ln|\Phi| J^a(\Phi/x).
\end{aligned}
$$

If we define a vectorial Jacobian $J^a(\Phi/x)$ of Φ and use the Laplace relation in Φ-space(here, $\triangle_\Phi := \partial^2/\partial\Phi^i\partial\Phi^i$):

$$
\begin{aligned}
\epsilon^{ij}J^a(\Phi/x) \quad : \quad &= \epsilon^{abc}\partial_b\Phi^i\partial_c\Phi^j, & (32) \\
\triangle_\Phi \ln|\Phi| &= 2\pi\delta^2(\vec{\Phi}),
\end{aligned}
$$

then, one gets

$$
(\nabla \times \vec{V})^a = 2\pi\delta^2(\vec{\Phi})J^a(\Phi/x). \tag{33}
$$

From the first equation of (30), one has

$$
\vec{A} + \lambda(x)^2\nabla \times \vec{B} = \frac{1}{g}\vec{V} \tag{34}
$$

where $\lambda(x) = 1/(\sqrt{2}g|\Phi|)$ is called effective penetrating length here in analogy to the typed-I superconductor. Taking curl of equation (34) yields

$$
\vec{B} - \lambda^2\nabla^2\vec{B} - 4g^2\lambda^4(\vec{\Phi}\partial_j\vec{\Phi})[\nabla B^j - \partial^j\vec{B}] = \frac{\phi_0}{2\pi}\nabla \times \vec{V}. \tag{35}
$$

where $\phi_0 = 2\pi/g$ is the unit quanta of vortex flux, as shown below.

We show that the vorticity in (33) can be written as the sum of delta-functions of the lines by using the formula[17]

$$
\delta^2(\vec{\Phi}) = \sum_k w_k(\vec{\Phi})\int_{L_k}ds\frac{\delta^3(\vec{r} - \vec{r}_k(s))}{D(\frac{\Phi}{u})\Sigma_k} \tag{36}
$$

$$
J(\frac{\Phi}{u})_{\Sigma_k} = \det\left[\frac{\partial\Phi^m}{\partial u^l}\right], (i, m = 1, 2) \tag{37}
$$

where L_k stands for the zero-lines $\vec{r}_k(s)$ of Φ given by $\Phi^1(x, y, z) = 0$,$\Phi^2(x, y, z) = 0$, $w_k(\vec{\Phi})$ is the winding number of map $\vec{\Phi} : x \to \Phi(x)$ around the singular line L_k , and Σ_k stands for the transverse section of L_k at $\vec{r}_k(s)$, with (u^1, u^2) being the surface parameters. One can show from (32) and (37) that

$$
\frac{\vec{J}(\frac{\Phi}{x})}{J(\frac{\Phi}{u})_{\Sigma_k}}\bigg|_{\vec{r}_k(s)} = \frac{d\vec{r}_k}{ds}
$$

11

which, combining with (36) and (33), leads to

$$\nabla \times \vec{V} = 2\pi \sum_k w_k(\vec{\Phi}) \int_{L_k} d\vec{r}_k \delta^3(\vec{r} - \vec{r}_k).$$

Integration of $\nabla \times \vec{V}$ over surface $\Sigma = \cup \Sigma_k$, the collection of all transverse section of L_k, gives the circulation quantization of $\vec{V}$: $\oint \vec{V} \cdot d\vec{l} = 2\pi n$, where $n = \sum_k w_k$ is a topological integer. Then, Eq.(35) becomes

$$\vec{B} - \lambda^2 \nabla^2 \vec{B} - 4g^2 \lambda^4 (\vec{\Phi} \partial_j \vec{\Phi})[\nabla B^j - \partial^j \vec{B}] = \phi_0 \sum_k w_k(\vec{\Phi}) \int_{L_k} d\vec{r}_k \delta^3(\vec{r} - \vec{r}_k), \qquad (38)$$

which is the generalized London's equation. The RHS of Eq.(38) represents the quantized ANO vortices, with unit flux quanta of $\phi_0 = 2\pi/g$. In the situation that $\Phi(x)$ varys very slowly over the space (analogy to London limit for typed II superconductor), Eq.(38) becomes

$$\vec{B} - \lambda^2 \nabla^2 \vec{B} = \phi_0 \sum_k w_k(\vec{\Phi}) \int_{L_k} d\vec{r}_k \delta^3(\vec{r} - \vec{r}_k).$$

which is the generalization of the London's equation $\vec{B} - \lambda^2 \nabla^2 \vec{B} = 0$ to vortex-state case[34]. Eq.(38) shows that the quantized flux for each vortex is given by $\phi_0 w_k(\vec{\Phi})$, the integer multiple of ϕ_0.

It should be pointed out that the RHS of Eq.(38) includes all topological contributions due to the vortices and is valid for many vortex-line situation. One can see from the above deduction that this topological term does not depend upon the dynamical details of the scalar $\Phi(x)$ and is due to the topological quantization of the flux of $\vec{B}$.

3.3 Concluding remark

We have shown that the effective dual AH action can follows from $SU(2)$ YM theory, based on the Faddeev-Niemi decomposition of $SU(2)$ gauge field and several assumptions. The analysis of the implication of these assumptions and the role of the resulted scalar field in the effective model suggests that one may reasonably look field Φ as the field of monopole and anti-monopole, whose superfluidity induces dual "repulsive effects" with respect to the "electric" field A_μ. This agrees with the analysis [23] of the magnetic charge operator. The mass generation for the "electric" field as well as the scalar can be due to quantum fluctuation of the local coset basis $\partial \mathbf{n}$. The approximate equality of masses between the "electric" field and the scalar is observed, being in agreement with the prediction of Ref.[28]. We also derive a generalized dual London equation with topologically quantized singular string for the static "electric" field from the dual AH model (23). We hope this will shed a light on the studies of the DS picture for color-confinement mechanism within the framework of YM theory.

In the opinion of this paper, the effective theory of YM theory in the far-infrared limit is that of the AH variable (A_μ, ϕ) at the classical level, with $\mathbf{n}$ being the quantum background. This differs with the opinion of Ref. [13] that the effective theory of infrared YM theory is that of $\mathbf{n}$

variable and that the theory vacuum is described by knotted string. However, in the absence of the external sources, it is favorable for vortex line in Eq.(38) to form closed or knotted string due to the tension of vortex tube. The analysis here implies that the vortex tube seems to be the "electric" knot, as was argued in [14].

We also note that the analysis of this paper is based on the on-shell CD (2), which ignores the marginal terms in the effective model (23). The inclusion of such a term can be done by appying the approach in this paper to the off-shell CD [13, 35]. The details of the relevant analysis will be presented in elsewhere.

Acknowledgements

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